Dynamics on $\hat{\mathbb{C}}$ and Julia Sets

Setting $f: \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ rat'l $\qquad f(z) = \dfrac{p(z)}{q(z)}$

Iterates $f^{\circ n}(z)$.

- Open set w/ predictable dynamics - Fatou set
- Closed set w/ chaotic dynamics - Julia set.

Goal: Julia sets have to have lots of self-similarity

Ex: $f(z) = z^2$

$$\left.\begin{array}{ll} |z|<1 & f^{\circ n}(z) \to 0 \\ |z|>1 & f^{\circ n}(z) \to \infty \end{array}\right\} \text{Fatou set}$$

$|z| = 1$ $\quad \cancel{f^{\circ n}(z)}$ $\quad$ - Julia set

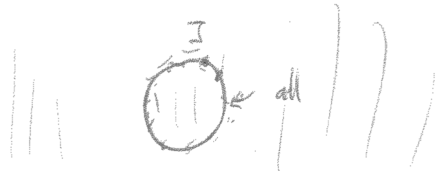
$z = 1 \quad f^{\circ n}(z) = 1$

$z = e^{2\pi i \varepsilon}$, ε irrational $\quad f^{\circ n}(z)$ dense in S^1!

- all have conf isomorphic nbhds

~~conf~~ (since $(f^{\circ n})'(z) \neq 0$ on S^1)

(z, J) conf isomorphic to (z', J)



- show printed examples here.

Julia set: $J(f)$ K cmpt, goal is to show

1) $\bigcup_{n=0}^{\infty} f^{-n}(z)$ dense in $J(f)$ (slightly more).

2) w/ finitely many exceptions, $\forall z \in J(f)$

$\exists$ dense set ~~s.t. $\forall z \in A$ $\exists$~~

Complex Analysis:

~~metric on $Hol(\hat{\mathbb{C}}, \hat{\mathbb{C}})$~~

$Hol(S, \hat{\mathbb{C}}) := \{ f: S \to \hat{\mathbb{C}},\ f \text{ holomorphic} \}$

Let ~~K_n b~~ $K_0 \subset K_1 \subset \dots \subset K_n$ be compact exhaustion of S.

~~$f, g \in Hol(S, \hat{\mathbb{C}})$~~

$\mu(y, y') := \min(d(y, y'), 1) \quad y, y' \in \hat{\mathbb{C}}$

~~$\mu(f, g) =$~~

round metric.

$f, g \in Hol(S, \hat{\mathbb{C}})$

$$\mu(f, g) = \sum_n \frac{1}{2^n} \max_{K_n} \{ d(f(x), g(x)) \}$$

Weierstrass conv. thm

$\Rightarrow$ $Hol(S, \hat{\mathbb{C}})$ is complete metric space w.r.t. μ.

~~Montel's Thm:~~

Def: $\mathcal{F} \subseteq Hol(S, \hat{\mathbb{C}})$ is normal if $\overline{\mathcal{F}}$ cmpct

~~(sequential compactness)~~

(every seq. has conv. ssubseq.)

Thm: Montel's Thm: $\mathcal{F} \subseteq Hol(S, \hat{\mathbb{C}})$

s.t. ~~$\forall f \in \mathcal{F}$~~ $\mathcal{F}(S) \subseteq \hat{\mathbb{C}} \setminus \{a, b, c\}$

Then $\mathcal{F}$ is normal.

Fatou + Julia (Fatou joke here).

$f: \hat{\mathbb{C}} \to \hat{\mathbb{C}}$

Fatou set $:= \{ p \mid \exists\, \mathcal{U} \overset{\text{open}}{\ni} p ;\ \{f^{\circ n}\}_{n=1}^{\infty}\big|_{\mathcal{U}}$ normal family$\}$.
$= F(f)$

Julia set $= J(f) := \hat{\mathbb{C}} \setminus F(f)$.

-sometimes, just write J, F.

Lemma: $f(J) = J$ $\qquad f(F) = F$

$\{f^{\circ n}\}_{n=1}^{\infty}\big|_{\mathcal{U}}$ normal $\Leftrightarrow$ $\{f^{\circ n}\}_{n=2}^{\infty}\big|_{\mathcal{U}}$ normal.

$\Rightarrow$ $f(J) \subseteq J$ + f surjective $\Rightarrow$ equality.
$f(F) \subseteq F$

Lemma: $J(f^{\circ k}) = J(f)$

Pf: Prove $F(f^{\circ k}) = F(f)$.

$\subseteq$ $\quad \overline{\{f^{\circ n}\}_{n=1}^{\infty}\big|_{\mathcal{U}}}$ cmpct $\Rightarrow$ $\overline{\{f^{\circ kn}\}_{n=1}^{\infty}}$ cmpct.
(seq. compactness)

$\supseteq$. $\quad \overline{\{f^{\circ kn}\}_{n=1}^{\infty}}$ cmpct $:= K$ $\Rightarrow$

$\overline{\{f^{\circ n}\}_{n=1}^{\infty}} \subseteq \underset{\text{cmpct.}}{K} \cup K \circ f \cup K \circ f^2 \cup \dots \cup K \circ f^{\circ(k-1)}$

Defn: Periodic orbit / cycle.

$$f: \quad z_0 \mapsto z_1 \mapsto z_2 \mapsto \dots \mapsto z_m = z_0.$$

If $z_1, \dots z_n$ distinct, ~~the~~ period is m.

$$\left(f^{\circ m}\right)'(z_0) = f'\left(f^{m-1}(z_0)\right)\left(f^{m-1}\right)'(z_0)$$
$$= f'(z_{m-1})\, f'(z_{m-2}) \dots f'(z_0) := \lambda.$$

λ is the multiplier of this orbit - same for each z_i.

- $|\lambda| < 1$ attracting ($|\lambda| = 0$ super attracting)
- $|\lambda| > 1$ repelling
- $|\lambda| = 1$ indifferent.

Prop: $z_0 \mapsto z_1 \mapsto \dots \mapsto z_n$ attracting $\{z_i\} \subseteq F(f)$

$z_0 \mapsto z_1 \mapsto \dots \mapsto z_n$ repelling $\{z_i\} \subseteq J(f)$.

Pf: Replace f by $f^{\circ m}$ - won't change J, F.

Attracting: $f(z_0) = z_0$ $\quad f'(z_0) = \lambda \quad |f'(z_0)| < 1.$

Taylors thm/fund thm calc.

$\Rightarrow |f(z) - z_0| < c|z - z_0| \quad c < 1$ for $z \in \mathcal{U} \ni z_0$ near z_0

$\Rightarrow f^{\circ n}(\mathcal{U}) \to z_0$ uniformly.

$\Rightarrow$ ~~$\mathcal{U}$~~ $z_0 \in F(f)$.

Repelling: $f^{\circ n_j} \to g$ in nbhd of z_0

Weierstrass $\Rightarrow (f^{\circ n_j})'(z_0) \to g'(z_0)$

$$\begin{array}{c} \parallel \\ \lambda^{n_j} \to \infty \end{array}, \text{ contradiction}$$

$\Rightarrow z_0 \in J(f)$

Lemma: $J(f)$ non-empty if $\deg(f) \geq 2$ ($=$ # preimages under f $= |\deg(p) - \deg(q)|$),

If $J(f)$ empty, $f^{\circ n_j} \to g$ on $\hat{\mathbb{C}}$

$\Rightarrow \deg(f^{\circ n_j}) = \deg(g)$ for $n_j >> 0$.

$$\begin{array}{c} \parallel \\ n_j \deg(f) \to \infty \end{array} \quad \lightning$$

~~Def: grand orbit of $z := \{z' \mid f^{\circ m}(z) = f^{\circ n}(z')$ some $n, m\}$.~~

~~Lemma: $\mathcal{E}(f) := \{z \mid$ grand orbit of z is finite$\}$~~

~~$|\mathcal{E}(f)| \leq 2$.~~

Main Thm: $z_* \in J(f)$, $z \in N$ open

Then $\bigcup_{n=0}^{\infty} f^{\circ n}(N) \supseteq J(f)$,

+ $\hat{\mathbb{C}} \setminus \bigcup_{n=0}^{\infty} f^{\circ n}(N)$ finite

Pf: $\mathcal{U} := \bigcup_{n=0}^{\infty} f^n(N)$

Claim: $|\mathcal{U}^c| \leq 2$, else have $f^{\circ n}: \mathcal{U} \to \mathcal{U}$

normal by Montel's thm

but $z \in J(f) \cap \mathcal{U}$ $\lightning$.

If $p \in \mathcal{U}^c$ know $f^{-1}(p) \subseteq \mathcal{U}^c$ (since $f(\mathcal{U}) \subseteq \mathcal{U}$)

$\Rightarrow$ f induces bijection on $\mathcal{U}^c$. - finite set

- w.t.s. consists of

~~Claim: $z_0 \in \mathcal{U}^c \Rightarrow f'(z_0) = 0 \quad (\Rightarrow z_0$~~ superattracting orbit.

$\cancel{\sigma} \Rightarrow \cancel{f(z_1) =} \quad \cancel{f(z_0) = f^{-1}(}$

$z_0 \overset{f}{\longmapsto} z_{\sigma(0)}$ $\Rightarrow f(z) - z_{\sigma(0)}$ zero of order ≥ 2 @ z_0

$z_1 \overset{f}{\longmapsto} z_{\sigma(1)}$

$\Rightarrow f'(z_0) = 0$

$\Rightarrow z_0$ in superattracting orbit $\blacksquare$.

So: ~~$\hat{\mathbb{C}} \setminus \mathcal{U}$ finite~~ $\mathcal{U}^c$ finite, $\hat{\mathbb{C}} \setminus \bigcup_{n=0}^{\infty} f^n(N)$ finite

consists only of superattracting orbits, i.e.

$$\hat{\mathbb{C}} \setminus \bigcup_{n=0}^{\infty} f^n(N) \subseteq F(f)$$

$$\Rightarrow \cancel{\hat{\mathbb{C}} \setminus \bigcup_{n=0}} \bigcup_{n=0}^{\infty} f^n(N) \supseteq J(f)$$

Cor: If $J(f)^\circ \neq \emptyset$, $J(f) = \hat{\mathbb{C}}$.

If $\exists N \subseteq J(f)$ open

$$\Rightarrow \bigcup_{n=0}^{\infty} f^{\circ n}(N) \subseteq J(f) \quad \text{- closed}$$

$$\begin{array}{c} \cup\!\shortmid \\ \hat{\mathbb{C}} \setminus \{\text{finite pts.}\} \end{array} \Rightarrow J(f) = \hat{\mathbb{C}}.$$

Cor. ~~4.3~~ Iterated Pre-images are Dense:

$z_0 \in J(f)$, $f^{-\infty}(z_0) := \{ z \in \hat{\mathbb{C}} \mid f^{\circ n}(z) = z_0 \quad n \geq 0 \}$.

dense in $J(f)$

Pf: If $\overline{f^{-\infty}(z_0)} \neq J(f)$

$\Rightarrow \exists \mathcal{U}$ open $\mathcal{U} \cap J(f) \neq \emptyset$, $f^{-\infty}(z_0) \cap \mathcal{U} = \emptyset$.

$\Rightarrow \bigcup_{i=0}^{\infty} f(\mathcal{U}) \not\ni z_0 \in J(f)$

$\cup\!\!\!\mid$

$J(f)$ $\lightning$

Cor: For generic $z \in J(f)$, $\{ f^{\circ n}(z) \mid n \in \mathbb{N} \}$ is dense in $J(f)$. } skip do if time

For each $j > 0$, $\exists$ finitely many $N_{j,k}$ of ~~radius~~ $\mathrm{diam}(N_{j,k}) < 1/j$

s.t. $J(f) \subseteq \bigcup N_{j,k}$

Set $\mathcal{U}_{j,k} = \bigcup_{n=0}^{\infty} f^{-n}(N_{j,k})$

$\overline{\mathcal{U}_{j,k} \cap J} = J$ by prev. i.e. $\mathcal{U}_{j,k} \cap J$ dense in J

$\Rightarrow$ Baire's thm, $\bigcap_{j,k} \mathcal{U}_{j,k} \cap J$ dense.

$\Rightarrow$ ~~$f^n(z)$~~ & $\forall z \in \bigcap_{j,k} (\mathcal{U}_{j,k} \cap J)$

$\Rightarrow f^n(z) \in N_{j,k}$ for some n, each j,k.

$\Rightarrow \{ f^n(z) \}$ dense.

Self - Similarity: $z, z' \in J(f)$

Say (J, z) conf. isomorphic to (J, z')

if $\exists N^{\ni z}, N'^{\ni z'}$ open

+ conf. isomorphism $N \to N'$

$J \cap N \mapsto J \cap N'$

~~$f(J) \ldots f(J \cap N) = \ldots$~~ $J \cap N'$

If: $f^n(z) = z'$ + $(f^n)'(z) \neq 0$

then (J, z), (J, z') conf. isomorphic.

~~Claim:~~ Prop. $z_0 \in J(f)$ $\{ z \mid (J, z)$ conf. isomorphic to $(J, z_0) \}$

dense unless for all backwards orbits

$z_0 \leftarrow z_1 \leftarrow z_2 \leftarrow z_3 \leftarrow \ldots$

some z_j, $j > 0$ is a crit pt. for f

Pf: $\{$crit pts$\} \cap \{ z \mid f^m(z) = z_0 \}$ finite.

$z_{j,1} \mapsto ? \ldots \mapsto z_0$

$\vdots$

$z_{j,n} \mapsto \ldots \mapsto z_0$

If $\exists$ backwards orbit hits crit. pt.

$z_0 \leftarrow z_1 \leftarrow \ldots \leftarrow z_{j,1}$

it does so in $< K$ steps.

So if $\exists 1$ $z_0 \leftarrow z_1 \leftarrow \ldots \leftarrow z_K$,

then $f^{-\infty}(z_K) \subseteq f^{-\infty}(z_0)$

dense. $(J, z_0) \sim (J, z)$ $\forall z \in f^{-\infty}(z_K)$

Say z_0 <u>exceptional</u> if this condition holds.

<u>Claim</u>: only finitely many exceptional pts.

<u>Pf</u>: $z_1 \dots z_j$ finitely many critical pts.

$$z_1 \mapsto z_{1,1} \mapsto z_{1,2} \mapsto \dots$$

$$z_2 \mapsto z_{2,1} \mapsto z_{2,2} \mapsto \dots$$

$$\vdots$$

$$z_j \mapsto z_{j,1} \mapsto z_{j,2} \mapsto \dots$$

Keep going unless
1) $z_{j,i}$ not exceptional (or)
2) $z_{j,i} = z_{k,\ell}$ for some k $\ell < i$
3) $z_{j,i} = z_{\ell,i}$ $\ell < i$

Note this procedure must list every exceptional pt.
& if z exceptional, $f^{-1}(z)$ - either exceptional or a critical pt.

<u>So</u> If stop, done.

If never stop. $z_1 \mapsto z_{1,1} \mapsto z_{2,1} \to \dots$

$$\vdots$$

$$z_\ell \mapsto z_{\ell,1} \mapsto z_{\ell,2} \mapsto \dots$$

go forever.

<u>Claim</u>: ~~Pf~~ For $i \gg 0$, $f^{-1}(z_{j,i}) = z_{j,i-1}$.

$f^{-1}(z_{j,i}) \subseteq$ this list, $\Rightarrow \not\exists$ would see it sooner or later, if $\not\exists$ we stop. ↯

$\Rightarrow f^{-1}(z_{j,i}) = z_{j,i-1}$

$\Rightarrow z_{j,i-1}$ all have ~~multiplicity~~ order ≥ 1 $\Rightarrow f'(z_{j,i-1}) = 0$. ↯

Prop: $J(f)$ connected or contains ∞ many conn components (uncountably).

If $J(f) = \bigcup_{i=1}^{n} U_i$ ← ∞ conn. comp.

$\Rightarrow f(U_i) = U_j$ i.e. f permutes U_i

$\Rightarrow f^{n!}(U_i) = U_i$

$\Rightarrow$

$\Rightarrow$ iterated preimages of $z \in U_i$ under $f^{n!}$ are not dense, $\lightning$

Q: When is $J(f)$ connected?

$f_c(z) := z^2 + c$.

$M_c := \{c \mid J(f_c) \text{ connected}\}$

= Mandlebrot set

- Say a few words here.

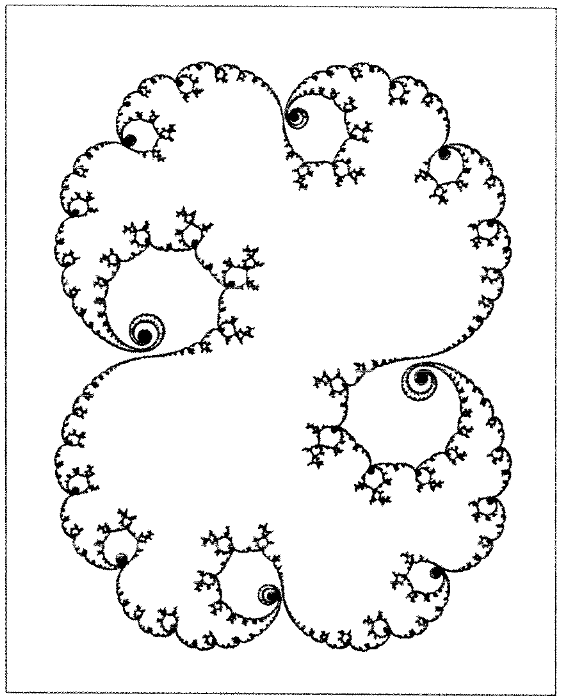


Figure 1a. A simple closed curve,
Julia set for $z \mapsto z^2 + (.99 + .14i)z$.

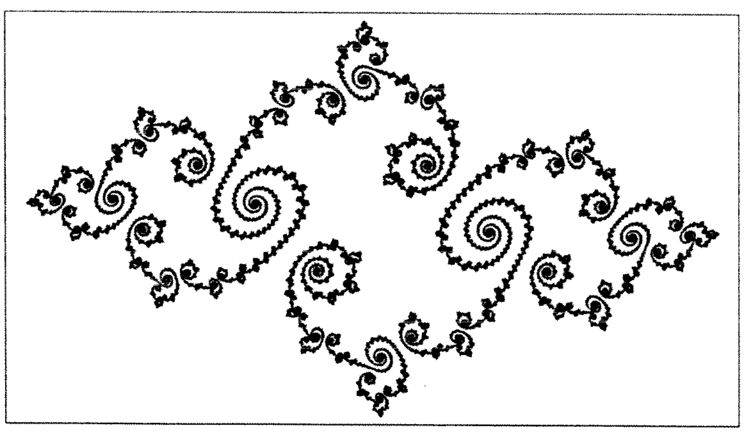


Figure 1b. A totally disconnected Julia set,
$z \mapsto z^2 + (-.765 + .12i)$.

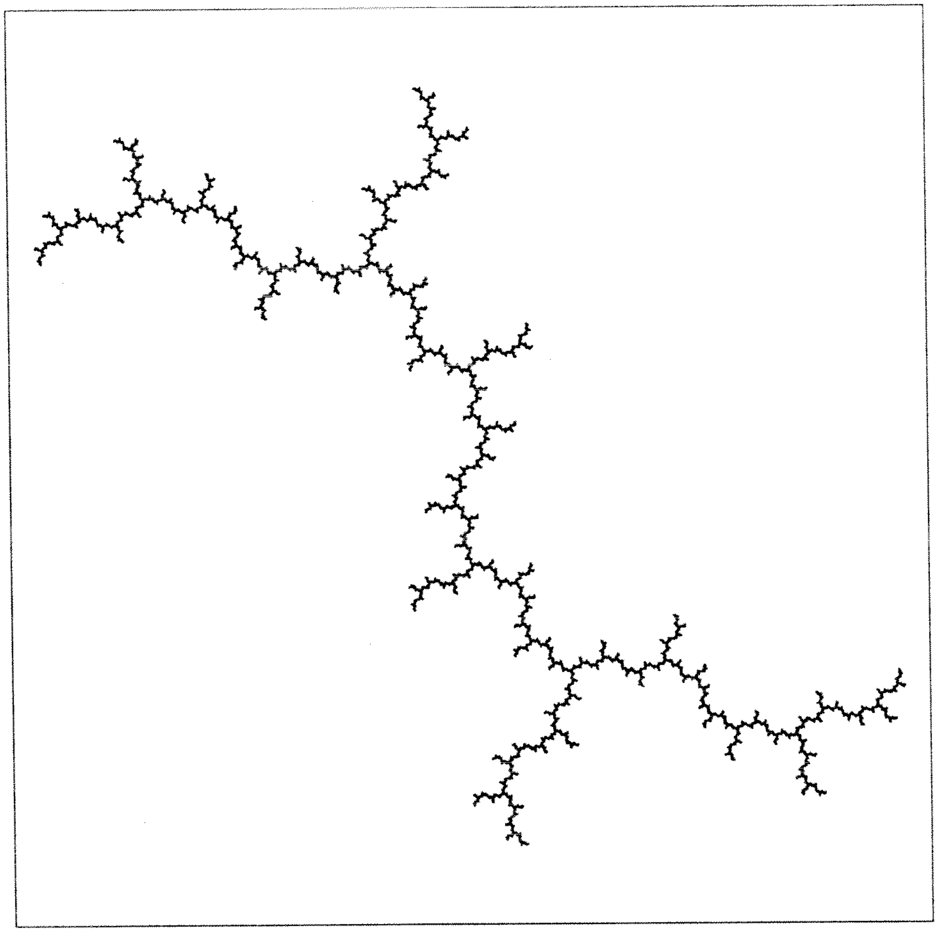


Figure 1c. A "dendrite", Julia set for $z \mapsto z^2 + i$.

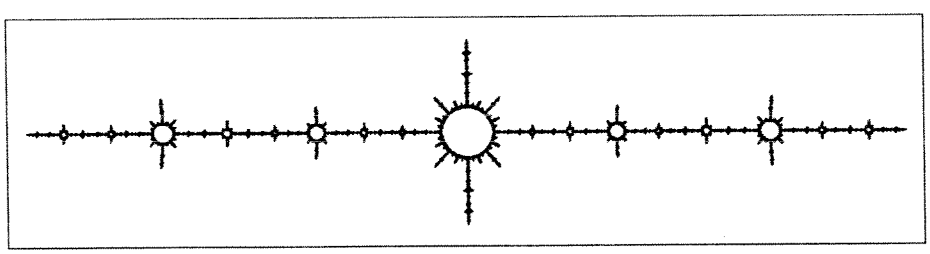


Figure 1d. Julia set for $z \mapsto z^2 - 1.75488\ldots$, the "airplane".